VIBRATIONS OF NONHOMOGENEOUS SPHERICALLY AEOLOTROPIC SPHERICAL SHELL

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ABSTRACT In this paper, two problems of radial and rotatory vibrations of non-homogeneous spherical shell of spherically aeolotropic material have been solved. The elastic constants c_{ij} are assumed to be of the form $c_{ij} = \mu_{ij}r^m$, μ_{ij}, m being constants and the density ρ is assumed to be of the form $\rho = \rho_0 r^s$, ρ_0 and s being constants.

The results for the homogeneous spherically isotropic case, those for the inhomogeneous spherically isotropic case of constant density and those for the non-homogeneous isotropic case are obtained from this paper as special cases.

INTRODUCTION

Sur (1964) has investigated the problem of vibration of inhomogeneous spherical shell of aeolotropic material of constant density.

Bose (1967) has recently solved the problem of torsional vibrations of non-homogeneous spherical and cylindrical shells of variable density and variable modulus of rigidity.

Chakravorty (1955) has investigated the problems of radial and rotatory vibrations of homogeneous spherical shell of spherically aeolotropic material and of uniform density. In the present paper, the corresponding problems for the non-homogeneous material are discussed. The nonhomogeneity of the shell is due to the variable density ρ and due to the variation of c_{ij}. The elastic constants c_{ij} are assumed to be $c_{ij} = \mu_{ij}r^m$, μ_{ij} and m being constants and r being the radius vector. The density of the material is assumed to be of the form $\rho = \rho_0 r^s$, ρ_0 and s being constants.

Lastly, the results for the homogeneous case (Chakravorty, 1955), those for the non-homogeneous case of constant density (Sur, 1964), and those for the non-homogeneous isotropic case (Bose, 1967) are obtained from the results of the present paper as special cases.

SOLUTION OF THE PROBLEM

Let $r = a$ and $r = b$ be the boundaries of the spherical shell. In spherical polar co-ordinates (r, θ, ϕ) the stress-strain relations in the non-homogeneous spherically aeolotropic shell in which $c_{ij} = \mu_{ij}r^m$ give

$$\left.\begin{aligned}
\widehat{rr} &= r^m\{\mu_{11}e_{rr}+\mu_{12}(e_{\theta\theta}+e_{\varphi\varphi})\} \\
\widehat{\theta\theta} &= r^m\{\mu_{12}e_{rr}+\mu_{22}e_{\theta\theta}+\mu_{23}e_{\varphi\varphi}\} \\
\widehat{\phi\phi} &= r^m\{\mu_{12}e_{rr}+\mu_{23}e_{\theta\theta}+\mu_{22}e_{\varphi\varphi}\} \\
\widehat{\theta\phi} &= \mu_{44}r^m e_{\theta\varphi} \\
\widehat{\phi r} &= \mu_{55}r^m e_{\varphi r} \\
\widehat{r\theta} &= \mu_{55}r^m e_{r\theta}
\end{aligned}\right\} \quad \ldots \quad (1)$$

where $\mu_{23} = \mu_{22}-2\mu_{44}$ and m, μ_{ij} are constants. The density ρ of the material of the shell is

$$\rho = \rho_0 r^s \quad \ldots \quad (2)$$

where ρ_0 and s are constants.

RADIAL VIBRATION

For the radial vibration of the spherical shell, the displacement components are assumed to be given by

$$u_r = Ue^{ipt}\,, \quad u_\theta = 0 = u_\varphi \quad \ldots \quad (3)$$

U being a function of r only and p being a constant.

The components of strain are therefore given by (Love, 1944, p. 56)

$$\left.\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{dU}{dr}\, e^{ipt} \\
e_{\theta\theta} &= \frac{u_r}{r}+\frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} = \frac{U}{r}\, e^{ipt} \\
e_{\varphi\varphi} &= \frac{u_r}{r}+\frac{u_\theta}{r}\cot\theta+\frac{1}{r\sin\theta}\,\frac{\partial u_\varphi}{\partial \phi} = \frac{U}{r}\, e^{ipt} \\
e_{\theta\varphi} &= \frac{1}{r\sin\theta}\,\frac{\partial u_\theta}{\partial \phi}+\frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \theta}-u_\varphi\cot\theta\right) = 0 \\
e_{\varphi r} &= \frac{1}{r\sin\theta}\,\frac{\partial u_r}{\partial \phi}+\frac{\partial u_\varphi}{\partial r}-\frac{u_\varphi}{r} = 0 \\
e_{r\theta} &= \frac{1}{r}\,\frac{\partial u_r}{\partial \theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r} = 0.
\end{aligned}\right\} \quad \ldots \quad (4)$$

(1) then gives the stress-components as

$$\left.\begin{aligned} \widehat{rr} &= r^m\left(\mu_{11}\frac{dU}{dr}+\frac{2\mu_{12}}{r}U\right)e^{ipt} \\ \widehat{\theta\theta} = \widehat{\phi\phi} &= r^m\left(\mu_{12}\frac{dU}{dr}+\frac{\mu_{22}+\mu_{23}}{r}U\right)e^{ipt} \\ \widehat{\theta\phi} = \widehat{\phi r} &= \widehat{r\theta} = 0. \end{aligned}\right\} \quad \ldots \quad (5)$$

Substitution of the stress-components from (5) and the density ρ from (2) into the only non-vanishing equation of motion (Love, 1944 p. 91), namely,

$$\frac{\partial\widehat{rr}}{\partial r}+\frac{1}{r}\frac{\partial\widehat{r\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\widehat{r\phi}}{\partial\phi}+\frac{1}{r}(2\widehat{rr}-\widehat{\theta\theta}-\widehat{\phi\phi}+\widehat{r\theta}\cot\theta) = \rho\frac{\partial^2 u_r}{\partial t^2}$$

yields

$$\frac{d^2U}{dr^2}+\frac{m+2}{r}\frac{dU}{dr}+\frac{2}{\mu_{11}}\{(m+1)\mu_{12}-\mu_{22}-\mu_{23}\}\frac{U}{r^2}+\frac{\rho_0 p^2}{\mu_{11}}r^{s-m}U = 0 \ldots \quad (6)$$

When $s \neq m-2$, using the transformations

$$\left.\begin{aligned} U &= r^{-\frac{m+1}{2}}V \\ \text{and} \quad z &= \frac{2k}{s-m+2}r^{\frac{s-m+2}{2}} \end{aligned}\right\} \quad \ldots \quad (7)$$

the equation (6) reduces to the Bessel's equation of the nth order, namely,

$$\frac{d^2V}{dz^2}+\frac{1}{z}\frac{dV}{dz}+\left(1-\frac{n^2}{z^2}\right)V = 0 \quad \ldots \quad (8)$$

where

$$\left.\begin{aligned} k &= p\sqrt{\frac{\rho_0}{\mu_{11}}} \\ l^2 &= \left(\frac{m+1}{2}\right)^2+\frac{2}{\mu_{11}}\{\mu_{22}+\mu_{23}-(m+1)\mu_{12}\} \\ \text{and} \quad n &= \frac{2l}{s-m+2}. \end{aligned}\right\} \quad \ldots \quad (9)$$

Solution of (8) is $$V = AJ_n(z)+BY_n(z) \qquad \dots \quad (10)$$

where $J_n(z)$ and $Y_n(z)$ are Bessel's functions of order n and of the first and second kind respectively and A, B are arbitrary constants. (7) then gives

$$U = r^{-\frac{m+1}{2}}\left[A\,J_n\left(\frac{nk}{l}\,r^{\frac{l}{n}}\right)+B\,Y_n\left(\frac{nk}{l}r^{\frac{l}{n}}\right)\right] \qquad \dots \quad (11)$$

The stress-component $\widehat{rr}$ is then given from (5) with the use of the recurrence relations

$$\left.\begin{aligned} J_n'(z) &= J_{n-1}(z)-\frac{n}{z}\,J_n(z)\\ Y_n'(z) &= Y_{n-1}(z)-\frac{n}{z}\,Y_n(z)\end{aligned}\right\} \qquad \dots \quad (12)$$

as

$$\widehat{rr} = r^{\frac{m-3}{2}}\left[A\left[kr^{\frac{l}{n}}\,\mu_{11}J_{n-1}(z)+\left\{2\mu_{12}-\left(l+\frac{m+1}{2}\right)\mu_{11}\right\}J_n(z)\right]\right.$$

$$\left.+B\left[kr^{\frac{l}{n}}\,\mu_{11}Y_{n-1}(z)+\left\{2\mu_{12}-\left(l+\frac{m+1}{2}\right)\mu_{11}\right\}Y_n(z)\right]\right]e^{ipt}. \qquad \dots \quad (13)$$

The boundary conditions on the boundaries $r = a$ and $r = b$ of the shell are $\widehat{rr} = 0$ on $r = a$ and $r = b$. (13) then gives by these boundary conditions,

$$A\left[ka^{\frac{l}{n}}\,\mu_{11}J_{n-1}(z_1)+\left\{2\mu_{12}-\left(l+\frac{m+1}{2}\right)\mu_{11}\right\}J_n(z_1)\right]$$

$$+B\left[ka^{\frac{l}{n}}\,\mu_{11}Y_{n-1}(z_1)+\left\{2\mu_{12}-\left(l+\frac{m+1}{2}\right)\mu_{11}\right\}Y_n(z_1)\right] = 0 \qquad \dots \quad (14)$$

and

$$A\left[kb^{\frac{l}{n}}\,\mu_{11}J_{n-1}(z_2)+\left\{2\mu_{12}-\left(l+\frac{m+1}{2}\right)\mu_{11}\right\}J_n(z_2)\right]$$

$$+B\left[kb^{\frac{l}{n}}\,\mu_{11}Y_{n-1}(z_2)+\left\{2\mu_{12}-\left(l+\frac{m+1}{2}\right)\mu_{11}\right\}Y_n(z_2)\right] = 0 \qquad \dots \quad (15)$$

where $$z_1 = [z]_{r=a} = \frac{nk}{l} a^{\frac{l}{n}}$$

and $$z_2 = [z]_{r=b} = \frac{nk}{l} b^{\frac{l}{n}}.$$

Elimination of A and B from (14) and (15) produces the frequency equation

$$\frac{2ka^{\frac{l}{n}}\mu_{11}J_{n-1}(z_1)+\{4\mu_{12}-(2l+m+1)\mu_{11}\}J_n(z_1)}{2ka^{\frac{l}{n}}\mu_{11}Y_{n-1}(z_1)+\{4\mu_{12}-(2l+m+1)\mu_{11}\}Y_n(z_1)}$$

$$=\frac{2kb^{\frac{l}{n}}\mu_{11}J_{n-1}(z_2)+\{4\mu_{12}-(2l+m+1)\mu_{11}\}J_n(z_2)}{2kb^{\frac{l}{n}}\mu_{11}Y_{n-1}(z_2)+\{4\mu_{12}-(2l+m+1)\mu_{11}\}Y_n(z_2)}. \qquad \ldots \quad (16)$$

When $s = m-2$, using the transformation $U = r^{-\frac{m+1}{2}} V$, the equation (6) reduces to

$$\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}-\lambda^2\frac{V}{r^2} = 0$$

where $$\lambda^2 = \frac{1}{\mu_{11}}\{2\mu_{22}+2\mu_{23}-2(m+1)\mu_{12}-\rho_0p^2\}+\left(\frac{m+1}{2}\right)^2.$$

The solution of this equation is $V = Ar^{\lambda}+Br^{-\lambda}$ where A, B are arbitrary constants.

Hence $$U = Ar^{\lambda-\frac{m+1}{2}}+Br^{-\lambda-\frac{m+1}{2}}$$

and $$\widehat{rr} = r^{\frac{m-3}{2}}\left[A\left\{2\mu_{12}+\mu_{11}\left(\lambda-\frac{m+1}{2}\right)\right\}r^{\lambda}\right.$$

$$\left.+B\left\{2\mu_{12}-\mu_{11}\left(\lambda+\frac{m+1}{2}\right)\right\}r^{-\lambda}\right]e^{ipt}.$$

The boundary conditions $\widehat{rr} = 0$ on $r = a$ and $r = b$ are satisfied if either $B = 0$ and $2\mu_{12}+\mu_{11}\left(\lambda-\frac{m+1}{2}\right) = 0$, or $A = 0$ and $2\mu_{12}-\mu_{11}\left(\lambda+\frac{m+1}{2}\right) = 0$.

The first or second condition will be satisfied according as $\frac{m+1}{2}-\frac{2\mu_{12}}{\mu_{11}}$ is positive or negative.

Now the frequency equation is

$$\rho_0 p^2 = \frac{2}{\mu_{11}}(\mu_{11}\mu_{12}+\mu_{11}\mu_{23}-2\mu_{12}^2)$$

which is identical with that for the case of constant density (Sur, 1964).

ROTATORY VIBRATION

For the rotatory vibration of the spherical shell the components of the displacement are assumed to be

$$u_r = 0 = u_\theta\,, \qquad u_\varphi = W \sin\theta . e^{ipt} \qquad \ldots \quad (17)$$

W being a function of r only.

Then the only non-vanishing component of stress is

$$\widehat{\phi r} = \mu_{55} r^m \left(\frac{dW}{dr} - \frac{W}{r}\right) \sin\theta .\ e^{ipt}. \qquad \ldots \quad (18)$$

Substitution of the only non-zero stress-component (18) into the only non-vanishing equation of motion, namely

$$\frac{\partial\widehat{\phi r}}{\partial r} + \frac{1}{r}\frac{\partial\widehat{\phi\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\widehat{\phi\phi}}{\partial\phi} + \frac{1}{r}(3\widehat{\phi r}+2\widehat{\phi\theta}\cot\theta) = \rho\frac{\partial^2 u_\varphi}{\partial t^2}$$

yields

$$\frac{d^2W}{dr^2} + \frac{2+m}{r}\frac{dW}{dr} - \frac{2+m}{r^2}W + \frac{\rho_0 p^2}{\mu_{55}} r^{s-m} W = 0 \qquad \ldots \quad (19)$$

When $s \neq m-2$, the transformations

$$\left.\begin{aligned} W &= r^{-\frac{m+1}{2}} X \\ y &= \frac{2h}{s-m+2} r^{\frac{s-m+2}{2}} \end{aligned}\right\} \qquad \ldots \quad (20)$$

and

reduce the equation (19) into

$$\frac{d^2X}{dy^2} + \frac{1}{y}\frac{dX}{dy} + \left(1-\frac{\nu^2}{y^2}\right)X = 0 \qquad \ldots \quad (21)$$

where $\qquad h = p\sqrt{\dfrac{\rho_0}{\mu_{55}}} \qquad$ and $\qquad \nu = \dfrac{m+3}{s-m+2}.$

A solution of (21) is $X = CJ_\nu(y) + DY_\nu(y)$ $\qquad \ldots \quad (22)$

where C and D are arbitrary constants. Hence from (20),

$$W = r^{-\frac{m+1}{2}} [CJ_\nu(y)+DY_\nu(y)] \qquad \dots \quad (23)$$

The stress-component $\widehat{\phi r}$ is then given from (18) with the help of the recurrence relations,

$$\left.\begin{aligned} J_\nu'(y) &= \frac{\nu}{y} J_\nu(y) - J_{\nu+1}(y) \\ \text{and} \quad Y_\nu'(y) &= \frac{\nu}{y} Y_\nu(y) - Y_{\nu+1}(y) \end{aligned}\right\} \qquad \dots \quad (24)$$

as

$$\widehat{\phi r} = -\tfrac{1}{2}\mu_{55} r^{\frac{m-3}{2}} (s-m+2) y \{CJ_{\nu+1}(y)+DY_{\nu+1}(y)\} \sin\theta . e^{ipt}. \qquad \dots \quad (25)$$

Since $\widehat{\phi r} = 0$ on the boundaries $r = a$ and $r = b$ of the spherical shell, therefore

$$CJ_{\nu+1}(y_1)+DY_{\nu+1}(y_1) = 0 \qquad \dots \quad (26)$$

and

$$CJ_{\nu+1}(y_2)+DY_{\nu+1}(y_2) = 0 \qquad \dots \quad (27)$$

where

$$y_1 = [y]_{r=a} = \frac{2h}{s-m+2} a^{\frac{s-m+2}{2}}$$

and

$$y_2 = [y]_{r=b} = \frac{2h}{s-m+2} b^{\frac{s-m+2}{2}}.$$

Elimination of C and D from (26) and (27) gives the frequency equation

$$\frac{J_{\nu+1}(y_1)}{Y_{\nu+1}(y_1)} = \frac{J_{\nu+1}(y_2)}{Y_{\nu+1}(y_2)} \qquad \dots \quad (28)$$

The case when $s = m-2$ can be treated exactly in the same way as in the case of radial vibration.

NUMERICAL RESULTS

Taking $m = 2$, $s = 1$ so that $\nu = 5$, the frequency equation (28) for the rotatory vibration becomes

$$\frac{Y_6(\omega)}{J_6(\omega)} = \frac{Y_6(\eta\omega)}{J_6(\eta\omega)} \qquad \dots \quad (29)$$

where $$\omega = 2p\sqrt{\frac{\rho_0 a}{\mu_{55}}} \quad \text{and} \quad \eta\omega = 2p\sqrt{\frac{\rho_0 b}{\mu_{55}}}$$

so that $$\eta = \sqrt{\frac{b}{a}} > 1.$$

It is known (Gray and Mathews, 1895, p. 242) that the q-th root, in order of magnitude of (29) is

$$w^{(q)} = \delta + \frac{\alpha}{\delta} + \frac{\beta-\alpha^2}{\delta^3} + \frac{\gamma-4\alpha\beta+2\alpha^3}{\delta^5} + \ldots\ldots$$

where $$\delta = \frac{q\pi}{\eta-1}, \quad \alpha = \frac{4(6)^2-1}{8\eta}, \quad \beta = \frac{4(4.6^2-1)(4.6^2-25)(\eta^3-1)}{3(8\eta)^3(\eta-1)}$$

and $$\gamma = \frac{32(4.6^2-1)(16.6^4-456.6^2+1073)(\eta^5-1)}{5(8\eta)^5(\eta-1)}.$$

Roots of (29) have been calculated for different values of η and are given in the following table :

	$\frac{a}{b}$	.25	.5	.75
	$\eta = \sqrt{\frac{b}{a}}$	2	1.414	1.154
$w^{(1)}$		5.2736	9.0803	21.1426
$w^{(2)}$		7.5591	15.9848	41.1775
$w^{(3)}$		10.3265	23.3129	61.4524

Corresponding to the qth root $\omega^{(q)}$, the frequency of vibration $p^{(q)}$ will be given by $p^{(q)} = \sqrt{\frac{\mu_{55}}{\rho_0 a}}\ \frac{\omega^{(q)}}{2}$.

DISCUSSIONS

The results for the homogeneous spherically aeolotropic shell (Chakravorty, 1955) may be obtained easily from the results of the present problem by putting $m = 0$ and $s = 0$.

Then from (9), $$n^2 = l^2 = \frac{1}{4}\left[1+\frac{8}{c_{11}}(c_{22}+c_{23}-c_{12})\right]$$

and from (21), $$\nu = 3/2.$$

So, from (16) the frequency equation for the radial vibration becomes

$$\frac{2c_{11}kaJ_{n-1}(ka)+(4c_{12}-c_{11}-2nc_{11})J_n(ka)}{2c_{11}kaY_{n-1}(ka)+(4c_{12}-c_{11}-2nc_{11})Y_n(ka)}$$

$$=\frac{2c_{11}kbJ_{n-1}(kb)+(4c_{12}-c_{11}-2nc_{11})J_n(kb)}{2c_{11}kbY_{n-1}(kb)+(4c_{12}-c_{11}-2nc_{11})Y_n(kb)}$$

and from (28) the frequency equation for the rotatory vibration becomes

$$J_{\frac{5}{2}}(ha)J_{-\frac{5}{2}}(hb) = J_{\frac{5}{2}}(hb)J_{-\frac{5}{2}}(ha) \quad [\because\ Y_{\frac{5}{2}}(x) = -J_{-\frac{5}{2}}(x)].$$

These agree with the results of chakravorty (1955).

The results of Sur (1964), may be obtained from the results of the present paper by putting $s = 0$.

The results for the non-homogeneous isotropic spherical shell (Bose, 1967 in which $\mu = \mu_0 r^m$ and $\rho = \rho_0 r^m$) are obtained from the results of the present paper by putting $\mu_{55} = \mu_0$ and $s = m$.

Then from (21), $\nu = \dfrac{m+3}{2}$ and from (28) the frequency equation for the torsional vibration becomes

$$J_{\frac{m+5}{2}}(ha)Y_{\frac{m+5}{2}}(hb) = J_{\frac{m+5}{2}}(hb)Y_{\frac{m+5}{2}}(ha)$$

which agree with the result of Bose (1967).

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REFERENCES

Bose, R. K., 1967, *Indian J. Phys.*, **41**, 533.

Chakravorty, J. G., 1955, *Bull. Cal. Math. Soc.*, **47**, 235.

Gray, A. and Mathews, G. B., 1895, *A Treatise on Bessel Functions*, Macmillan and Co.

Love, A. E. H., 1944, *A Treatise on the Mathematical Theory of Elasticity*, 4th edition, Dover Publication.

Sur, S. P. 1964, *Ind. Jour. Mech. Maths.*, **2**, 6.